Navorsings- en Oorsigartikels

Statistiese analise gebaseer op kwaternioonnormaalvariate

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UITTREKSEL

Die kwaternioonnormaalverdeling word afgelei en enkele eienskappe word uitgelig. Die maksimum-aanneemlikheidsberamingsprosedure in die kwaternioongeval word ondersoek en die gevolgtrekking gemaak dat die beramingsprosedure telkens vergemaklik word indien die onbekende parameters van die geassosieerde reële waarskynlikheidsdigtheidsfunksie beraam word. Deur hierdie beramers dan as die komponente van die kwaternioonparameter te beskou, word die kwaternioonberamer verkry. By wyse van 'n voorbeeld word aandag gegee aan 'n toetskriterium wat in die kwaternioonmodel gebruik kan word.

ABSTRACT

Statistical analysis based on quaternion normal random variables

The quaternion normal distribution is derived and a number of characteristics are highlighted. The maximum likelihood estimation procedure in the quaternion case is examined and the conclusion is reached that the estimation procedure is simplified if the unknown parameters of the associated real probability density function are estimated. The quaternion estimator is then obtained by regarding these estimators as the components of the quaternion estimator. By means of a example attention is given to a test criterium which can be used in the quaternion model.

1. INLEIDING

Sedert die ontwikkeling van kwaternione het verskeie navorsers die eienskappe en toepassingsmoontlikhede van kwaternione ondersoek. Die gebruik van kwaternione is vandag by heelwat wetenskaplikes onbekend, alhoewel daar vanaf die sestigerjare opnuut 'n belangstelling in kwaternione ontstaan het. Die ontwikkeling en ontdekking van steeds nuwe probleme in die kwantummeganika het kwaternione weer 'n regmatige plek in die algebraïese manipulasie van ingewikkelde en omvattende fisiese sisteme gegee. Hier het Roux en Rautenbach (1984) 'n statistiese benadering tot die toets van Peres (1979), om te onderskei tussen reële, komplekse en kwaternioonkwantumteorieë, voorgestel.

Omdat kwaternione relatief onbekend is, is baie min aandag gegee aan die ontwikkeling van die teorie van die kwaternioonverdelingsleer. Kabe het waarskynlik die tegnieke van Goodman (1963) toegepas om 'n kwaternioonnormaalverdeling af te lei, want daar word telkens verwys na 'n artikel, „Generalized normal multivariate models'', wat aangebied is vir publikasie, maar tot op hede nog nie verskyn het nie. Kabe (1976) gee egter 'n paar basiese eienskappe van so 'n kwaternioonnormaalverdeling. Hieruit blyk dit duidelik dat hoewel die werk in die algemeen algebraïes korrek is, hy 'n aannamefout begaan het, naamlik dat hy die feit dat kwaternioonvermenigvuldiging nie-kommutatief is, veronagsaam het. Hierdie fundamentele eienskap van kwaternione wat geïgnoreer is, lewer heelwat komplikasies op, onder andere dat die gebruik van die spoor, en spesifiek die gebruik daarvan in die uitdrukking van die kwaternioonwaarskynlikheidsdigtheidsfunksie, nie toelaatbaar is nie.

In paragraaf 3 word in die afleiding van die kwaternioonnormaalverdeling gebruik gemaak van die representasieteorie, dit is die gebruik van matrikse in die plek van die elemente van 'n abstrakte delingsalgebra. Om 'n representasie te konstrueer wat delingsalgebra-elemente oordra na matrikse, is dit nodig om te weet wanneer versamelings van matrikse 'n delingsalgebra vorm. Hierdie aspek word in Rautenbach en Roux (1983) bespreek. Rautenbach en Roux definieer verder 'n aantal wiskunde begrippe en toon aan dat die kwaternione 'n assosiatiewe delingsalgebra oor die liggaam van reële getalle vorm wat gerepresenteer word deur matrikse met 'n spesifieke struktuur. In paragraaf 4 word aandag gegee aan enkele belangrike eienskappe van die kwaternioonnormaalverdeling. Die maksimum-aanneemlikheidsberamers in die kwaternioongeval word in paragraaf 5 gevind. Ten slotte word gekyk na 'n toetskriterium wat in die kwaternioonmodel gebruik kan word.

2. ALGEBRAÏESE RESULTATE VIR KWATERNIONE

In hierdie paragraaf word sekere algebraïese resultate genoem wat benodig word vir die afleiding van die kwaternioonnormaalverdeling (vergelyk Rautenbach en Roux, 1983).

DEFINISIE 2.1

Laat $R_{st} : 4\times 4 = \begin{bmatrix} a_{1st} & -a_{2st} & -a_{3st} & -a_{4st} \\ a_{2st} & a_{1st} & -a_{4st} & a_{3st} \\ a_{3st} & a_{4st} & a_{1st} & -a_{2st} \\ a_{4st} & -a_{3st} & a_{2st} & a_{1st} \end{bmatrix}$

en $q_{st} = a_{1st} + ia_{2st} + ja_{3st} + ka_{4st}$, met a_{1st}, a_{2st}, a_{3st} en a_{4st} reële getalle. Definieer nou die reële matriks $R : 4p\times 4p$ en die matriks $Q : p\times p$ met kwaternioon-inskrywings respektiewelik as

$$R : 4p\times 4p = [R_{st}] \quad \text{en} \quad Q : p\times p = [q_{st}].$$

DEFINISIE 2.2

Definieer 'n afbeelding $f : M_{4p}(\mathcal{R}) \rightarrow M_p(\mathcal{Q})$ deur

$$f(R : 4p\times 4p) = Q : p\times p$$

vir alle

$$R \in M_{4p}(\mathcal{R}) = \{R : 4p\times 4p | R_{st} \in R,\ s, t = 1, \ldots, p\}$$

en

$$Q \in M_p(\mathcal{Q}) = \{Q : p\times p | q_{st} \in Q,\ s, t = 1, \ldots, p\}.$$

OPMERKING 2.1

Indien $R : 4p\times 4p \in M_{4p}(\mathcal{R})$ en $f(R) = Q \in M_p(\mathcal{Q})$, sal dit kortweg aangedui word deur $R \simeq Q$.

STELLING 2.1

Veronderstel $R : 4p\times 4p \in M_{4p}(\mathcal{R})$, $Q : p\times p \in M_p(\mathcal{Q})$ en $R \simeq Q$. Dan is R nie-singulier, met inverse

$$R^{-1} : 4p\times 4p = [r^{st}] = \begin{bmatrix} a_1^{st} & -a_2^{st} & -a_3^{st} & -a_4^{st} \\ a_2^{st} & a_1^{st} & -a_4^{st} & a_3^{st} \\ a_3^{st} & a_4^{st} & a_1^{st} & -a_2^{st} \\ a_4^{st} & -a_3^{st} & a_2^{st} & a_1^{st} \end{bmatrix}$$

as en slegs as Q nie-singulier is met inverse

$$Q^{-1} : p\times p = [q^{st}] = [a_1^{st} + ia_2^{st} + ja_3^{st} + ka_4^{st}] .$$

STELLING 2.2

Veronderstel $R : 4p\times 4p \in M_{4p}(\mathcal{R})$, $Q : p\times p \in M_p(\mathcal{Q})$ en $R \simeq Q$. Indien R simmetries is, dan is

$$\det R = (\det Q)^4 .$$

STELLING 2.3

Laat

$$\underset{\sim}{g}_0 : 4p\times 1 = [b_{11}, b_{21}, b_{31}, b_{41}, \ldots, b_{1p}, b_{2p}, b_{3p}, b_{4p}]'$$
$$\underset{\sim}{g} : p\times 1 = [q_1, \ldots, q_p]'$$

met $q_s = b_{1s} + ib_{2s} + jb_{3s} + kb_{4s}$, $s = 1, \ldots, p$. Veronderstel dat $R : 4p\times 4p \in M_{4p}(\mathcal{R})$, $Q : p\times p \in M_p(\mathcal{Q})$ en $R \simeq Q$.

Laar R simmetries wees sodat Q kwaternioon-Hermities is.

Dan volg:

$$\underset{\sim}{g}_0' R \underset{\sim}{g}_0 = \bar{\underset{\sim}{g}}' Q \underset{\sim}{g} .$$

OPMERKING 2.2

Veronderstel $R : 4p\times 4p \in M_{4p}(\mathcal{R})$, $Q : p\times p \in M_p(\mathcal{Q})$ en $R \simeq Q$. Dan is R simmetries positief-definiet as en selgs as Q kwaternioon-Hermities positief-definiet is.

3. DIE p-VERANDERLIKE KWATERNIOONNORMAALVERDELING

In hierdie paragraaf word die p-veranderlike kwaternioonnormaalverdeling gedefinieer en die ooreenstemmende waarskynlikheidsdigtheidsfunksie (wdf.) word afgelei.

DEFINISIE 3.1

$$\text{Laat } \underset{\sim}{Z} : p\times 1 = [Z_1, \ldots, Z_p]' = \begin{bmatrix} X_{11} + iX_{21} + jX_{31} + kX_{41} \\ X_{12} + iX_{22} + jX_{32} + kX_{42} \\ \vdots \\ X_{1p} + iX_{2p} + jX_{3p} + kX_{4p} \end{bmatrix}$$

'n vektor van kwaternioonvariate wees met geassosieerde vektor van reële variate

$$\underset{\sim}{Z}_0 : 4p\times 1 = [X_{11}, \ldots, X_{1p}, X_{21}, \ldots, X_{2p}, X_{31}, \ldots, X_{3p}, X_{41}, \ldots, X_{4p}]' = [\underset{\sim}{X}_1', \underset{\sim}{X}_2', \underset{\sim}{X}_3', \underset{\sim}{X}_4']'.$$

Dan het $\underset{\sim}{Z}$ 'n p-veranderlike kwaternioonnormaalverdeling indien $\underset{\sim}{Z}_0$ 'n 4p-veranderlike reële normaalverdeling besit.

Kabe (1976) gee aandag aan die geval waar $\underset{\sim}{Z}_0$ 'n heel besondere 4p-veranderlike reële normaalverdeling besit, naamlik die geval waar

$$\underset{\sim}{\mu}_0 \equiv E(\underset{\sim}{Z}_0) = [\underset{\sim}{\mu}_{x_1}', \underset{\sim}{\mu}_{x_2}', \underset{\sim}{\mu}_{x_3}', \underset{\sim}{\mu}_{x_4}']' = [E(\underset{\sim}{X}_1'), E(\underset{\sim}{X}_2'), E(\underset{\sim}{X}_3'), E(\underset{\sim}{X}_4')]'$$

en

$$\Sigma_0 : 4p\times 4p = \text{kov}(\underset{\sim}{Z}_0, \underset{\sim}{Z}_0') = E[(\underset{\sim}{Z}_0 - \underset{\sim}{\mu}_0)(\underset{\sim}{Z}_0 - \underset{\sim}{\mu}_0)'] = \tfrac{1}{4} \begin{bmatrix} \Sigma_1 & -\Sigma_2 & -\Sigma_3 & -\Sigma_4 \\ \Sigma_2 & \Sigma_1 & -\Sigma_4 & \Sigma_3 \\ \Sigma_3 & \Sigma_4 & \Sigma_1 & -\Sigma_2 \\ \Sigma_4 & -\Sigma_3 & \Sigma_2 & \Sigma_1 \end{bmatrix}$$

waar $\Sigma_1 : p\times p$ 'n reële simmetriese matriks is en Σ_2, Σ_3 en Σ_4 reële skeef-simmetriese matrikse is. Bostaande kovariansiestruktuur vir $\underset{\sim}{Z}_0$ is die onderskeidende kenmerk van die p-veranderlike kwaternioonnormaalverdeling en hierdie struktuur impliseer die volgende:

$$\Sigma_1 = \begin{bmatrix} \sigma_1^2 & \eta_{12}\sigma_1\sigma_2 & \cdots & \eta_{1p}\sigma_1\sigma_p \\ \eta_{12}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \eta_{2p}\sigma_2\sigma_p \\ \vdots & \vdots & & \vdots \\ \eta_{1p}\sigma_1\sigma_p & \eta_{2p}\sigma_2\sigma_p & \cdots & \sigma_p^2 \end{bmatrix},$$

$$\text{waar } \eta_{st} = \frac{\sigma_{1st}}{\sigma_s\sigma_t},\quad s, t = 1, \ldots, p,\ s < t$$

$$\Sigma_2 = \begin{bmatrix} 0 & \alpha_{12}\sigma_1\sigma_2 & \alpha_{13}\sigma_1\sigma_3 & \cdots & \alpha_{1p}\sigma_1\sigma_p \\ -\alpha_{12}\sigma_1\sigma_2 & 0 & \alpha_{23}\sigma_2\sigma_3 & \cdots & \alpha_{2p}\sigma_2\sigma_p \\ \vdots & \vdots & & & \vdots \\ -\alpha_{1p}\sigma_1\sigma_p & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

$$\text{waar } \alpha_{st} = \frac{\sigma_{2st}}{\sigma_s\sigma_t},\quad s, t = 1, \ldots, p,\ s < t$$

$$\Sigma_3 = \begin{bmatrix} 0 & \beta_{12}\sigma_1\sigma_2 & \beta_{13}\sigma_1\sigma_3 & \ldots & \beta_{1p}\sigma_1\sigma_p \\ -\beta_{12}\sigma_1\sigma_2 & 0 & \beta_{23}\sigma_2\sigma_3 & \ldots & \beta_{2p}\sigma_2\sigma_p \\ \vdots & \vdots & & & \vdots \\ -\beta_{1p}\sigma_1\sigma_p & \ldots & \ldots & \ldots & 0 \end{bmatrix},$$

$$\text{waar } \beta_{st} = \frac{\sigma_{3st}}{\sigma_s\sigma_t},\quad s, t = 1, \ldots, p,\ s < t$$

$$\Sigma_4 = \begin{bmatrix} 0 & \gamma_{12}\sigma_1\sigma_2 & \gamma_{13}\sigma_1\sigma_3 & \ldots & \gamma_{1p}\sigma_1\sigma_p \\ -\gamma_{12}\sigma_1\sigma_2 & 0 & \gamma_{23}\sigma_2\sigma_3 & \ldots & \gamma_{2p}\sigma_2\sigma_p \\ \vdots & \vdots & & & \vdots \\ -\gamma_{1p}\sigma_1\sigma_p & \ldots & \ldots & \ldots & 0 \end{bmatrix},$$

$$\text{waar } \gamma_{st} = \frac{\sigma_{4st}}{\sigma_s\sigma_t},\quad s, t = 1, \ldots, p,\ s < t\,.$$

Ten einde die resultate wat in paragraaf 2 genoem is, te gebruik word die geassosieerde vektor van reële variate se komponente herrangskik, naamlik in die volgorde

$$\underset{\sim}{Z}_0^* : 4p \times 1 = [X_{11}, X_{21}, X_{31}, X_{41}, \ldots, X_{1p}, X_{2p}, X_{3p}, X_{4p}]'.$$

Die komponente van $\underset{\sim}{\mu}_0$ en Σ_0 word net soos $\underset{\sim}{Z}_0^*$ herrangskik om $\underset{\sim}{\mu}_0^* : 4p \times 1$ en $\Sigma_0^* : 4p \times 4p$ onderskeidelik te lewer, en dus word die wdf. van $\underset{\sim}{Z}_0^*$ gegee deur

$$f_{\underset{\sim}{Z}_0^*}(\underset{\sim}{z}_0^*) = (2\pi)^{-\frac{1}{2}(4p)}\{\det \Sigma_0^*\}^{-\frac{1}{2}} \text{eks}\{-\tfrac{1}{2}(\underset{\sim}{z}_0^* - \underset{\sim}{\mu}_0^*)'\Sigma_0^{*-1}(\underset{\sim}{z}_0^* - \underset{\sim}{\mu}_0^*)\}$$

vir

$$\underset{\sim}{z}_0^* \in B_0^* = \{\underset{\sim}{z}_0^* = [x_{11}, x_{21}, x_{31}, x_{41}, \ldots, x_{1p}, x_{2p}, x_{3p}, x_{4p}]' ;\ -\infty < x_{1s}, x_{2s}, x_{3s}, x_{4s} < \infty,\ s = 1, 2, \ldots, p\}\,.$$

Omdat $f_{\underset{\sim}{Z}_0}(\underset{\sim}{z}_0) = f_{\underset{\sim}{Z}_0^*}(\underset{\sim}{z}_0^*)$ vir alle $\underset{\sim}{z}_0$ en ooreenstemmende $\underset{\sim}{z}_0^*$ is dit dus toelaatbaar om $\underset{\sim}{Z}_0^*$ as geassosieerde vektor van reële variate te gebruik wanneer $\underset{\sim}{Z}$ se wdf. afgelei word.

Die reële kovariansiematriks $\Sigma_0^* : 4p \times 4p$, van $\underset{\sim}{Z}_0^* = [X_{11}, X_{21}, X_{31}, X_{41}, \ldots, X_{1p}, X_{2p}, X_{3p}, X_{4p}]'$ word soos volg gegee

$$\Sigma_0^* : 4p \times 4p = [\Sigma_{0st}^*]$$

waar elke Σ_{0st}^* 'n 4×4-matriks is van die vorm

$$\Sigma_{0ss}^* : 4 \times 4 = \tfrac{1}{4}\begin{bmatrix} \sigma_s^2 & 0 & 0 & 0 \\ 0 & \sigma_s^2 & 0 & 0 \\ 0 & 0 & \sigma_s^2 & 0 \\ 0 & 0 & 0 & \sigma_s^2 \end{bmatrix} \quad \text{vir } s = 1, 2, \ldots, p$$

en

$$\Sigma_{0st}^* : 4 \times 4 = \tfrac{1}{4}\begin{bmatrix} \eta_{st} & -\alpha_{st} & -\beta_{st} & -\gamma_{st} \\ \alpha_{st} & \eta_{st} & -\gamma_{st} & \beta_{st} \\ \beta_{st} & \gamma_{st} & \eta_{st} & -\alpha_{st} \\ \gamma_{st} & -\beta_{st} & \alpha_{st} & \eta_{st} \end{bmatrix} \sigma_s\sigma_t \quad \text{vir } s, t = 1, \ldots, p \text{ en } s \neq t\,.$$

Bostaande kovariansiestruktuur is 'n belangrike kenmerk van die p-veranderlike kwaternioonnormaalverdeling, want Σ_0^* is 'n matriks met elemente soortgelyk soos gegee in definisie 2.1. Dus kan gesê word dat $\Sigma_0^* \in M_{4p}(\mathcal{R})$ oftewel $4\Sigma_0^* \in M_{4p}(\mathcal{R})$. Vanuit definisie 2.2 is die duidelik dat bewerkings op $M_{4p}(\mathcal{R})$ met matrikse van die vorm $4\Sigma_0^*$ isomorf is aan bewerkings op $M_p(Q)$. Dus is $M_p(Q)$ 'n versameling van matrikse van die vorm $\Sigma : p \times p = [\sigma_{st}]$, waar

$$(3.1)\quad \sigma_{st} = \begin{cases} (\eta_{st} + i\alpha_{st} + j\beta_{st} + k\gamma_{st})\sigma_s\sigma_t\,, & \text{vir } s \neq t \\ \sigma_s^2 & \text{vir } s = t\,. \end{cases}$$

Dus volg dat $4\Sigma_0^* \simeq \Sigma$ (vergelyk opmerking 2.1).

Vervolgens word die wdf. van $\underset{\sim}{Z} : p \times 1$ gegee, waar $\underset{\sim}{Z}$ 'n p-veranderlike kwaternioonnormaalverdeling besit.

STELLING 1.3.1

Laat $\underset{\sim}{Z} : p \times 1$ 'n vektor van variate wees wat p-veranderlik kwaternioonnormaal verdeel is soos gegee in definisie 3.1 met $E(\underset{\sim}{Z}) = \underset{\sim}{\mu}$ en $\Sigma \equiv [\sigma_{st}]$ soos gegee in (3.1).

Die wdf. van $\underset{\sim}{Z}$ word gegee deur

$$(3.2)\quad f_{\underset{\sim}{Z}}(\underset{\sim}{z}) = 2^{2p}\pi^{-2p}(\det\Sigma)^{-2}\text{eks}\{-2\overline{(\underset{\sim}{z} - \underset{\sim}{\mu})}'\Sigma^{-1}(\underset{\sim}{z} - \underset{\sim}{\mu})\}$$

waar

$$\underset{\sim}{z} \in B = \{\underset{\sim}{z} = [z_1, \ldots, z_p]' : z_s = x_{1s} + ix_{2s} + jx_{3s} + kx_{4s}\ ;\ -\infty < x_{1s}, x_{2s}, x_{3s}, x_{4s} < \infty,\ s = 1, 2, \ldots, p\}\,.$$

BEWYS

Die wdf. van $\underset{\sim}{Z}_0^*$ word gegee deur

$$f_{\underset{\sim}{Z}_0^*}(\underset{\sim}{z}_0^*) = 2^{-2p}\pi^{-2p}\{\det\Sigma_0^*\}^{-\frac{1}{2}}\text{eks}\{-\tfrac{1}{2}(\underset{\sim}{z}_0^* - \underset{\sim}{\mu}_0^*)' \bullet \Sigma_0^{*-1}(\underset{\sim}{z}_0^* - \underset{\sim}{\mu}_0^*)\}$$

vir $\underset{\sim}{z}_0^* \in B_0^*$.

Nou is $4\Sigma_0^* \simeq \Sigma$. Omdat Σ_0^* simmetries positief-definiet is, volg dit uit opmerking 2.2 dat Σ kwaternioon-Hermities positief-definiet is, sodat Σ in besonder ook nie-singulier is. Verder lewer stelling 2.2 dat

$$\det(4\Sigma_0^*) = (\det\Sigma)^4$$

en dus

$$\det \Sigma_0^* = 4^{-4p}(\det\Sigma)^4$$

dit wil sê

$$(3.3)\quad (\det\Sigma_0^*)^{-\frac{1}{2}} = 4^{2p}(\det\Sigma)^{-2}.$$

Uit stelling 2.1 volg dat $\frac{1}{4}\Sigma_0^{*-1} \simeq \Sigma^{-1}$ sodat uit stelling 2.3 volg dat

$$(3.4)\quad 2(\underset{\sim}{z}_0^* - \underset{\sim}{\mu}_0^*)'\tfrac{1}{4}\Sigma_0^{*-1}(\underset{\sim}{z}_0^* - \underset{\sim}{\mu}_0^*) = 2\overline{(\underset{\sim}{z} - \underset{\sim}{\mu})}'\Sigma^{-1}(\underset{\sim}{z} - \underset{\sim}{\mu}).$$

Uitdrukkings (3.3) en (3.4) tesame met die bostaande uitdrukking vir $f_{\underset{\sim}{Z}_0^*}(\underset{\sim}{z}_0^*)$ lewer

$$f_{\underset{\sim}{Z}}(\underset{\sim}{z}) = 2^{2p}\pi^{-2p}(\det\Sigma)^{-2}\text{eks}\{-2\overline{(\underset{\sim}{z} - \underset{\sim}{\mu})}'\Sigma^{-1}(\underset{\sim}{z} - \underset{\sim}{\mu})\}$$

vir $\underset{\sim}{z} \in B$.

OPMERKING 3.1

(i) Indien 'n vektor van variate $\underset{\sim}{Z} : p \times 1$ kwaternioonnormaal verdeel is met wdf. (3.2) sal dit aangedui word deur

$$\underline{Z} \sim QMN(p;\underline{\mu},\Sigma)$$

waar $\underline{\mu}: p \times 1$ dan die verwagte waarde en $\Sigma: p \times p$ die kovariansiematriks van $\underline{Z}$ is.

(ii) Die wdf. van die twee-veranderlike kwaternioonnormaalverdeling word verkry deur $p = 2$ in (3.2) te stel.

4. EIENSKAPPE VAN DIE KWATERNIOONNORMAALVERDELING

In hierdie paragraaf word aandag gegee aan enkele eienskappe van die kwaternioonnormaalverdeling.

STELLING 4.1

$$\phi_{\underline{Z}}(\underline{t}) = \text{eks}[\tfrac{i}{2}(\bar{\underline{\mu}}'\underline{t} + \bar{\underline{t}}'\underline{\mu}) - \tfrac{1}{8}\bar{\underline{t}}'\Sigma\underline{t}]$$

waar $\underline{t}: p \times 1 = \underline{t}_1 + i\underline{t}_2 + j\underline{t}_3 + k\underline{t}_4$ 'n kwaternioonvektor van draers is.

BEWYS

Volgens definisie is

$$\phi_{\underline{Z}}(\underline{t}) = E[\text{eks}\,\tfrac{i}{2}(\bar{\underline{Z}}'\underline{t} + \bar{\underline{t}}'\underline{Z})]$$

waar $\underline{t}: p \times 1 = \underline{t}_1 + i\underline{t}_2 + j\underline{t}_3 + k\underline{t}_4$ 'n kwaternioonvektor van draers en i die gewone komplekse imaginêre eenheid is.

Dus volg dat

$$\phi_{\underline{Z}}(\underline{t}) = \int_B 2^{2p}\pi^{-2p}(\det\Sigma)^{-2}\text{eks}\{\tfrac{i}{2}(\bar{\underline{z}}'\underline{t} + \bar{\underline{t}}'\underline{z}) - 2(\overline{\underline{z}-\underline{\mu}})'\Sigma^{-1}(\underline{z}-\underline{\mu})\}\,d\underline{z}$$

met $B = \{\underline{z} = [z_1, \ldots, z_p]' : z_s = x_{1s} + ix_{2s} + jx_{3s} + kx_{4s};\ -\infty < x_{1s}, x_{2s}, x_{3s}, x_{4s} < \infty,\ s = 1, 2, \ldots, p\}$

die variasiegebied van $\underline{Z}$.

Die eksponent in bostaande uitdrukking kan geskryf word as

$$-2\{\bar{\underline{z}}'\Sigma^{-1}\underline{z} - \bar{\underline{z}}'\Sigma^{-1}\underline{\mu} - \bar{\underline{\mu}}'\Sigma^{-1}\underline{z} + \bar{\underline{\mu}}'\Sigma^{-1}\underline{\mu} - \tfrac{i}{4}\bar{\underline{z}}'\underline{t} - \tfrac{i}{4}\bar{\underline{t}}'\underline{z}\}$$

$$= -2\{(\overline{\underline{z} - [\underline{\mu} + \tfrac{i}{4}\Sigma\underline{t}]})'\Sigma^{-1}(\underline{z} - [\underline{\mu} + \tfrac{i}{4}\Sigma\underline{t}])\} + \tfrac{i}{2}(\bar{\underline{\mu}}'\underline{t} + \bar{\underline{t}}'\underline{\mu}) - \tfrac{1}{8}\bar{\underline{t}}'\Sigma\underline{t}$$

sodat

$$\phi_{\underline{Z}}(\underline{t}) = \text{eks}\{\tfrac{i}{2}(\bar{\underline{\mu}}'\underline{t} + \bar{\underline{t}}'\underline{\mu}) - \tfrac{1}{8}\bar{\underline{t}}'\Sigma\underline{t}\}\int_B 2^{2p}\pi^{-2p}(\det\Sigma)^{-2} \bullet \text{eks}\{-2(\overline{\underline{z} - [\underline{\mu} + \tfrac{i}{4}\Sigma\underline{t}]})'\Sigma^{-1}(\underline{z} - [\underline{\mu} + \tfrac{i}{4}\Sigma\underline{t}])\}d\underline{z}.$$

Omdat

$$2^{2p}\pi^{-2p}(\det\Sigma)^{-2}\text{eks}\{-2(\overline{\underline{z} - [\underline{\mu} + \tfrac{i}{4}\Sigma\underline{t}]})'\Sigma^{-1}(\underline{z} - [\underline{\mu} + \tfrac{i}{4}\Sigma\underline{t}])\}$$

'n p-veranderlike kwaternioon-wdf. is, volg

$$\phi_{\underline{Z}}(\underline{t}) = \text{eks}[\tfrac{i}{2}(\bar{\underline{\mu}}'\underline{t} + \bar{\underline{t}}'\underline{\mu}) - \tfrac{1}{8}\bar{\underline{t}}'\Sigma\underline{t}].$$

OPMERKING 4.1

Die eerste- en tweede-orde momente van die betrokke vektor van variate se komponente kan maklik verkry word deur middel van differensiasie van hierdie karakteristieke funksie.

STELLING 4.2

Laat $\underline{Z} \sim QMN(p;\underline{\mu},\Sigma)$ dan is

$\underline{Y} = C(\underline{Z} - \underline{\mu}) \sim QMN(p;\underline{0},I_p)$ indien C só gekies word dat $C\Sigma\bar{C}' = I_p$ en C 'n simplektiese matriks is.

BEWYS

Laat C sodanig wees dat $C\Sigma\bar{C}' = I_p$. Dan word die karakteristieke funksie gegee deur

$$\begin{aligned}\phi_{\underline{Y}}(\underline{t}) &= E\,\text{eks}\{\tfrac{i}{2}(\bar{\underline{Y}}'\underline{t} + \bar{\underline{t}}'\underline{Y})\}\\ &= \text{eks}\{-\tfrac{i}{2}(\bar{\underline{\mu}}'\bar{C}'\underline{t} + \bar{\underline{t}}'C\underline{\mu})\}\,E\,\text{eks}\{\tfrac{i}{2}(\bar{\underline{Z}}'\bar{C}'\underline{t} + \bar{\underline{t}}'C\underline{Z})\}\\ &= \text{eks}\{-\tfrac{i}{2}(\bar{\underline{\mu}}'\bar{C}'\underline{t} + \bar{\underline{t}}'C\underline{\mu})\}\,\text{eks}\{\tfrac{i}{2}(\bar{\underline{\mu}}'\bar{C}'\underline{t} + \bar{\underline{t}}'C\underline{\mu}) - \tfrac{1}{8}\bar{\underline{t}}'C\Sigma\bar{C}'\underline{t}\}\\ &= \text{eks}\{-\tfrac{1}{8}\bar{\underline{t}}'\underline{t}\},\end{aligned}$$

waaruit die verlangde resultaat volg.

STELLING 4.3

Laat $\underline{Z} \sim QMN(p;\underline{\mu},\Sigma)$ dan is

$4(\overline{\underline{Z} - \underline{\mu}})'\Sigma^{-1}(\underline{Z} - \underline{\mu})$ verdeel soos 'n χ^2-verdeling met 4p vryheidsgrade.

BEWYS

Laat $\underline{Y} = C(\underline{Z} - \underline{\mu})$ waar $C\Sigma\bar{C}' = I_p$ met C simplekties. Uit stelling (4.2) volg dan dat $\underline{Y} \sim QMN(p;\underline{0},I_p)$ sodat

$$\begin{aligned}\bar{\underline{Y}}'\underline{Y} &= \sum_{s=1}^{p} \bar{Y}_s Y_s\\ &= \tfrac{1}{4}\sum_{s=1}^{p}(4Y_{s1}^2 + 4Y_{s2}^2 + 4Y_{s3}^2 + 4Y_{s4}^2)\end{aligned}$$

waar elke $Y_{sr}(r = 1, 2, 3, 4) \sim N(0;\tfrac{1}{4})$ sodat $4Y_{sr}^2(r = 1, 2, 3, 4) \sim \chi_1^2$. Hieruit volg dus dat

$$\begin{aligned}4\bar{\underline{Y}}'\underline{Y} &= 4(\overline{\underline{Z} - \underline{\mu}})'\bar{C}'C(\underline{Z} - \underline{\mu})\\ &= 4(\overline{\underline{Z} - \underline{\mu}})'\Sigma^{-1}(\underline{Z} - \underline{\mu}) \sim \chi_{4p}^2.\end{aligned}$$

5. MAKSIMUM-AANNEEMLIKHEIDSBERAMERS

Die beraming van die onbekende parameters volgens die metode van momente is voor-die-handliggend, maar dit is tog nodig om na te gaan of die metode van masimum aanneemlikheid met sy bekende voordeliger eienskappe in die kwaternioon geval gebruik kan word. As gevolg van die feit dat kwaternioonvermenigvuldiging nie-kommutatief is, is dit moeilik om die onbekende parameters volgens die metode van maksimum aanneemlikheid direk in die kwaternioonruimte te beraam. Om dié rede word die beramers van die onbekende parameters van die geassosieerde wdf. eers bepaal en dan word hierdie beramers só saamgestel dat dit die beramers vir die onbekende kwaternioonparameters is. Ter wille van die vereenvoudiging van die algebraïese uitdrukkings word die aandag bepaal by die tweeveranderlike kwaternioonnormaalverdeling, dit wil sê die geassosieerde agtveranderlike normaalverdeling.

STELLING 5.1

Laat $\underline{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X_1 + iX_2 + jX_3 + kX_4 \\ Y_1 + iY_2 + jY_3 + kY_4 \end{bmatrix} \sim QMN(2;\underline{\mu},\Sigma)$

waar

$$\underline{\mu} = \begin{bmatrix} \mu_{z_1} \\ \mu_{z_2} \end{bmatrix}$$

en

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \xi_{11} + i\xi_{12} + j\xi_{13} + k\xi_{14} \\ \xi_{11} - i\xi_{12} - j\xi_{13} - k\xi_{14} & \sigma_2^2 \end{bmatrix}$$

waar $\xi_{11} = \eta_{12}\sigma_1\sigma_2$, $\xi_{12} = \alpha_{12}\sigma_1\sigma_2$, $\xi_{13} = \beta_{12}\sigma_1\sigma_2$ en $\xi_{14} = \gamma_{12}\sigma_1\sigma_2$. Die maksimum-aanneemlikheidsberamers word gegee deur

(5.1) $\hat{\mu}_{z_1} = \frac{1}{n}\sum_{s=1}^{n} z_{1s}$

(5.2) $\hat{\mu}_{z_2} = \frac{1}{n}\sum_{s=1}^{n} z_{2s}$

(5.3) $\hat{\sigma}_1^2 = \frac{1}{n}[\sum_{s=1}^{n} \bar{z}_{1s}z_{1s} - \frac{1}{n}(\sum_{s=1}^{n} \bar{z}_{1s})(\sum_{s=1}^{n} z_{1s})]$

(5.4) $\hat{\sigma}_2^2 = \frac{1}{n}[\sum_{s=1}^{n} \bar{z}_{2s}z_{2s} - \frac{1}{n}(\sum_{s=1}^{n} \bar{z}_{2s})(\sum_{s=1}^{n} z_{2s})]$

(5.5) $\hat{\xi}_{11} + i\hat{\xi}_{12} + j\hat{\xi}_{13} + k\hat{\xi}_{14} = \frac{1}{n}[\sum_{s=1}^{n} z_{1s}\bar{z}_{2s} - \frac{1}{n}(\sum_{s=1}^{n} z_{1s})(\sum_{s=1}^{n} \bar{z}_{2s})]$

(5.6) $\hat{\xi}_{11} - i\hat{\xi}_{12} - j\hat{\xi}_{13} - k\hat{\xi}_{14} = \overline{(\hat{\xi}_{11} + i\hat{\xi}_{12} + j\hat{\xi}_{13} + k\hat{\xi}_{14})}$.

BEWYS

Die geassosieerde reële wdf. word gegee deur

$$f_{\underline{Z}_0^*}(\underline{z}_0^*) = \pi^{-4}\, 2^{-4}\{\det \Sigma_0^*\}^{-\frac{1}{2}} \text{eks}\{-\tfrac{1}{2}(\underline{z}_0^* - \underline{\mu}_0^*)'\Sigma_0^{*-1} \bullet (\underline{z}_0^* - \underline{\mu}_0^*)\}$$

waar

$$\Sigma_0^* = \tfrac{1}{4}\begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 & \xi_{11} & -\xi_{12} & -\xi_{13} & -\xi_{14} \\ 0 & \sigma_1^2 & 0 & 0 & \xi_{12} & \xi_{11} & -\xi_{14} & \xi_{13} \\ 0 & 0 & \sigma_1^2 & 0 & \xi_{13} & \xi_{14} & \xi_{11} & -\xi_{12} \\ 0 & 0 & 0 & \sigma_1^2 & \xi_{14} & -\xi_{13} & \xi_{12} & \xi_{11} \\ \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} & \sigma_2^2 & 0 & 0 & 0 \\ -\xi_{12} & \xi_{11} & \xi_{14} & -\xi_{13} & 0 & \sigma_2^2 & 0 & 0 \\ -\xi_{13} & -\xi_{14} & \xi_{11} & \xi_{12} & 0 & 0 & \sigma_2^2 & 0 \\ -\xi_{14} & \xi_{13} & -\xi_{12} & \xi_{11} & 0 & 0 & 0 & \sigma_2^2 \end{bmatrix}$$

Die oplossing van die vergelykings

$\frac{\partial}{\partial_{\mu_v}} \prod_{s=1}^{n} \ell n\, f_{\underline{Z}_{0s}^*}(\underline{z}_{0s}^*) = 0,\ v = 1, \ldots, 9$ lewer

$$\frac{1}{n}\sum_{s=1}^{n} x_{1s} - \hat{\mu}_1 = \frac{1}{n\hat{\sigma}_2^2}\sum_{s=1}^{n} [(y_{1s} - \hat{\mu}_5)\hat{\xi}_{11} - (y_{2s} - \hat{\mu}_6)\hat{\xi}_{12} - \hat{\xi}_{13}(y_{3s} - \hat{\mu}_7) - \hat{\xi}_{14}(y_{4s} - \hat{\mu}_8)],$$

$$\frac{1}{n}\sum_{s=1}^{n} x_{2s} - \hat{\mu}_2 = \frac{1}{n\hat{\sigma}_2^2}\sum_{s=1}^{n} [(y_{1s} - \hat{\mu}_5)\hat{\xi}_{12} + (y_{2s} - \hat{\mu}_6)\hat{\xi}_{11} - (y_{3s} - \hat{\mu}_7)\hat{\xi}_{14} + (y_{4s} - \hat{\mu}_8)\hat{\xi}_{13}],$$

$$\frac{1}{n}\sum_{s=1}^{n} x_{3s} - \hat{\mu}_3 = \frac{1}{n\hat{\sigma}_2^2}\sum_{s=1}^{n} [(y_{2s} - \hat{\mu}_6)\hat{\xi}_{14} + (y_{3s} - \hat{\mu}_7)\hat{\xi}_{11} - (y_{4s} - \hat{\mu}_8)\hat{\xi}_{12} + \hat{\xi}_{13}(y_{1s} - \hat{\mu}_5)],$$

$$\frac{1}{n}\sum_{s=1}^{n} x_{4s} - \hat{\mu}_4 = \frac{1}{n\hat{\sigma}_2^2}\sum_{s=1}^{n} [(y_{1s} - \hat{\mu}_5)\hat{\xi}_{14} - (y_{2s} - \hat{\mu}_6)\hat{\xi}_{13} + (y_{3s} - \hat{\mu}_7)\hat{\xi}_{12} + (y_{4s} - \hat{\mu}_8)\hat{\xi}_{11}],$$

$$\frac{1}{n}\sum_{s=1}^{n} y_{1s} - \hat{\mu}_5 = \frac{1}{n\hat{\sigma}_1^2}\sum_{s=1}^{n} [(x_{1s} - \hat{\mu}_1)\hat{\xi}_{11} + \hat{\xi}_{12}(x_{2s} - \hat{\mu}_2) + \hat{\xi}_{13}(x_{3s} - \hat{\mu}_3) + \hat{\xi}_{14}(x_{4s} - \hat{\mu}_4)],$$

$$\frac{1}{n}\sum_{s=1}^{n} y_{2s} - \hat{\mu}_6 = \frac{1}{n\hat{\sigma}_1^2}\sum_{s=1}^{n} [-(x_{1s} - \hat{\mu}_1)\hat{\xi}_{12} + (x_{2s} - \hat{\mu}_2)\hat{\xi}_{11} + (x_{3s} - \hat{\mu}_3)\hat{\xi}_{14} - (x_{4s} - \hat{\mu}_4)\hat{\xi}_{13}],$$

$$\frac{1}{n}\sum_{s=1}^{n} y_{3s} - \hat{\mu}_7 = \frac{1}{n\hat{\sigma}_1^2}\sum_{s=1}^{n} [-(x_{2s} - \hat{\mu}_2)\hat{\xi}_{14} + (x_{3s} - \hat{\mu}_3)\hat{\xi}_{11} + (x_{4s} - \hat{\mu}_4)\hat{\xi}_{12} - \hat{\xi}_{13}(x_{1s} - \hat{\mu}_1)] \text{ en}$$

$$\frac{1}{n}\sum_{s=1}^{n} y_{4s} - \hat{\mu}_8 = \frac{1}{n\hat{\sigma}_1^2}\sum_{s=1}^{n} [-(x_{1s} - \hat{\mu}_1)\hat{\xi}_{14} + (x_{2s} - \hat{\mu}_2)\hat{\xi}_{13} - (x_{3s} - \hat{\mu}_3)\hat{\xi}_{12} + (x_{4s} - \hat{\mu}_8)\hat{\xi}_{11}].$$

Die enigste oplossing vir hierdie stel vergelykings is

$$\frac{1}{n}\sum_{s=1}^{n} x_{1s} - \hat{\mu}_1 = \frac{1}{n}\sum_{s=1}^{n} x_{2s} - \hat{\mu}_2 = \frac{1}{n}\sum_{s=1}^{n} x_{3s} - \hat{\mu}_3 = \frac{1}{n}\sum_{s=1}^{n} x_{4s} - \hat{\mu}_4 = \frac{1}{n}\sum_{s=1}^{n} y_{1s} - \hat{\mu}_5 = \frac{1}{n}\sum_{s=1}^{n} y_{2s} - \hat{\mu}_6 = \frac{1}{n}\sum_{s=1}^{n} y_{3s} - \hat{\mu}_7 = \frac{1}{n}\sum_{s=1}^{n} y_{4s} - \hat{\mu}_8 = 0$$

sodat

$$\hat{\mu}_1 = \frac{1}{n}\sum_{s=1}^{n} x_{1s}\ ;\ \hat{\mu}_2 = \frac{1}{n}\sum_{s=1}^{n} x_{2s}\ ;\ \hat{\mu}_3 = \frac{1}{n}\sum_{s=1}^{n} x_{3s}\ ;$$

$$\hat{\mu}_4 = \frac{1}{n}\sum_{s=1}^{n} x_{4s}\ ;\ \hat{\mu}_5 = \frac{1}{n}\sum_{s=1}^{n} y_{1s}\ ;\ \hat{\mu}_6 = \frac{1}{n}\sum_{s=1}^{n} y_{2s}\ ;$$

$$\hat{\mu}_7 = \frac{1}{n}\sum_{s=1}^{n} y_{3s}\ ;\ \hat{\mu}_8 = \frac{1}{n}\sum_{s=1}^{n} y_{4s}\ .$$

Oplossing van die vergelyking

$\frac{\partial}{\partial \sigma_1^2} \prod_{s=1}^{n} \ell n\, f_{\underline{Z}_0^*}(\underline{z}_0^*) = 0$, lewer

(5.8)
$$-2n\,\hat{\sigma}_2^2\{\hat{\sigma}_1^2\hat{\sigma}_2^2 - (\hat{\xi}_{11}^2 + \hat{\xi}_{12}^2 + \hat{\xi}_{13}^2 + \hat{\xi}_{14}^2)\} = 2(\hat{\sigma}_1^2\hat{\sigma}_2^2 - (\hat{\xi}_{11}^2 + \hat{\xi}_{12}^2 + \hat{\xi}_{13}^2 + \hat{\xi}_{14}^2)) \sum_{s=1}^{n} \{(y_{1s} - \hat{\mu}_5)^2 + (y_{2s} - \hat{\mu}_6)^2 + (y_{3s} - \hat{\mu}_7)^2 + (y_{4s} - \hat{\mu}_8)^2\} - 2\hat{\sigma}_2^2 \sum_{s=1}^{n} (\underline{z}_{0s}^* - \hat{\underline{\mu}}_0^*)'\hat{\bar{\Sigma}}_0^*(\underline{z}_{0s}^* - \hat{\underline{\mu}}_0^*),$$

met $\bar{\Sigma}_0^*$ die toegevoegde matriks.

Oplossing van die vergelyking

$\frac{\partial}{\partial \sigma_2^2} \prod_{s=1}^{n} \ell n\, f_{\underline{Z}_0^*}(\underline{z}_0^*) = 0$, lewer

(5.9)
$$-2n\,\hat{\sigma}_1^2\{\hat{\sigma}_1^2\hat{\sigma}_2^2 - (\hat{\xi}_{11}^2 + \hat{\xi}_{12}^2 + \hat{\xi}_{13}^2 + \hat{\xi}_{14}^2)\} = 2(\hat{\sigma}_1^2\hat{\sigma}_2^2 - (\hat{\xi}_{11}^2 + \hat{\xi}_{12}^2 + \hat{\xi}_{13}^2 + \hat{\xi}_{14}^2)) \sum_{s=1}^{n} \{(x_{1s} - \hat{\mu}_1)^2 + (x_{2s} - \hat{\mu}_2)^2 + (x_{3s} - \hat{\mu}_3)^2 + (x_{4s} - \hat{\mu}_4)^2\} - 2\hat{\sigma}_1^2 \sum_{s=1}^{n} (\underline{z}_{0s}^* - \hat{\underline{\mu}}_0^*)'\hat{\bar{\Sigma}}_0^*(\underline{z}_{0s}^* - \hat{\underline{\mu}}_0^*).$$

Oplossing van die vergelyking

$\frac{\partial}{\partial \xi_{11}} \prod_{s=1}^{n} \ell n\, f_{\underline{Z}_{0s}^*}(\underline{z}_{0s}^*) = 0$, lewer

(5.10)

$$4n\,\hat\sigma_1^2\hat\sigma_2^2\,\hat\xi_{11} - 4n\,\hat\xi_{11}(\hat\xi_{11}^2+\hat\xi_{12}^2+\hat\xi_{13}^2+\hat\xi_{14}^2)$$
$$= -4(\hat\sigma_1^2\hat\sigma_2^2-(\hat\xi_{11}^2+\hat\xi_{12}^2+\hat\xi_{13}^2+\hat\xi_{14}^2))\sum_{s=1}^{n}\{(x_{1s}-\hat\mu_1)(y_{1s}-\hat\mu_5)$$
$$+ (x_{2s}-\hat\mu_2)(y_{2s}-\hat\mu_6)+(x_{3s}-\hat\mu_3)(y_{3s}-\hat\mu_7)$$
$$+ (x_{4s}-\hat\mu_4)(y_{4s}-\hat\mu_8)\}+4\hat\xi_{11}\sum_{s=1}^{n}(\underline{z}_{0s}^*-\hat{\underline{\mu}}_0^*)'\hat{\hat{\Sigma}}_0^*(\underline{z}_{0s}^*-\hat{\underline{\mu}}_0^*).$$

Oplossing van die vergelyking

$$\frac{\partial}{\partial\xi_{12}}\prod_{s=1}^{n}\ell n\, f_{\underline{Z}_{0s}^*}(\underline{z}_{0s}^*) = 0\ ,\ \text{lewer}$$

(5.11)

$$4n\,\hat\sigma_1^2\hat\sigma_2^2\,\hat\xi_{12} - 4n\,\hat\xi_{12}(\hat\xi_{11}^2+\hat\xi_{12}^2+\hat\xi_{13}^2+\hat\xi_{14}^2)$$
$$= 4(\hat\sigma_1^2\hat\sigma_2^2-(\hat\xi_{11}^2+\hat\xi_{12}^2+\hat\xi_{13}^2+\hat\xi_{14}^2))\sum_{s=1}^{n}\{(x_{1s}-\hat\mu_1)(y_{2s}-\hat\mu_6)$$
$$+ (x_{3s}-\hat\mu_3)(y_{4s}-\hat\mu_8)-(x_{2s}-\hat\mu_2)(y_{1s}-\hat\mu_5)$$
$$- (x_{4s}-\hat\mu_4)(y_{3s}-\hat\mu_7)\}+4\hat\xi_{12}\sum_{s=1}^{n}(\underline{z}_{0s}^*-\hat{\underline{\mu}}_0^*)'\hat{\hat{\Sigma}}_0^*(\underline{z}_{0s}^*-\hat{\underline{\mu}}_0^*).$$

Oplossing van die vergelyking

$$\frac{\partial}{\partial\xi_{13}}\prod_{s=1}^{n}\ell n\, f_{\underline{Z}_{0s}^*}(\underline{z}_{0s}^*) = 0\ ,\ \text{lewer}$$

(5.12)

$$4n\,\hat\sigma_1^2\hat\sigma_2^2\,\hat\xi_{13} - 4n\,\hat\xi_{13}(\hat\xi_{11}^2+\hat\xi_{12}^2+\hat\xi_{13}^2+\hat\xi_{14}^2)$$
$$= 4[\hat\sigma_1^2\hat\sigma_2^2-(\hat\xi_{11}^2+\hat\xi_{12}^2+\hat\xi_{13}^2+\hat\xi_{14}^2)]\sum_{s=1}^{n}\{(x_{1s}-\hat\mu_1)(y_{3s}-\hat\mu_7)$$
$$+ (x_{4s}-\hat\mu_4)(y_{2s}-\hat\mu_7)-(x_{2s}-\hat\mu_2)(y_{4s}-\hat\mu_8)$$
$$- (x_{3s}-\hat\mu_3)(y_{1s}-\hat\mu_5)\}+4\hat\xi_{13}\sum_{s=1}^{n}(\underline{z}_{0s}^*-\hat{\underline{\mu}}_0^*)'\hat{\hat{\Sigma}}_0^*(\underline{z}_{0s}^*-\hat{\underline{\mu}}_0^*).$$

Oplossing van die vergelyking

$$\frac{\partial}{\partial\xi_{14}}\prod_{s=1}^{n}\ell n\, f_{\underline{Z}_{0s}^*}(\underline{z}_{0s}^*) = 0\ ,\ \text{lewer}$$

(5.13)

$$4n\,\hat\sigma_1^2\hat\sigma_2^2\,\hat\xi_{14} - 4n\,\hat\xi_{14}(\hat\xi_{11}^2+\hat\xi_{12}^2+\hat\xi_{13}^2+\hat\xi_{14}^2)$$
$$= 4[\hat\sigma_1^2\hat\sigma_2^2-(\hat\xi_{11}^2+\hat\xi_{12}^2+\hat\xi_{13}^2+\hat\xi_{14}^2)]\sum_{s=1}^{n}\{(x_{1s}-\hat\mu_1)(y_{4s}-\hat\mu_8)$$
$$+ (x_{2s}-\hat\mu_2)(y_{3s}-\hat\mu_7)-(x_{3s}-\hat\mu_3)(y_{2s}-\hat\mu_6)$$
$$- (x_{4s}-\hat\mu_4)(y_{1s}-\hat\mu_5)\}+4\hat\xi_{14}\sum_{s=1}^{n}(\underline{z}_{0s}^*-\hat{\underline{\mu}}_0^*)'\hat{\hat{\Sigma}}_0^*(\underline{z}_{0s}^*-\hat{\underline{\mu}}_0^*).$$

Oplossing van vergelykings (5.8) – (5.13) lewer

$$\hat\xi_{11} = \tfrac{1}{n}[\sum_{s=1}^{n}\sum_{r=1}^{4}(x_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}x_{rs})(y_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}y_{rs})]\ ,$$

$$\hat\xi_{12} = \hat\xi_{11}[\sum_{s=1}^{n}\{(x_{2s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{2s})(y_{1s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{1s})$$
$$+ (x_{4s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{4s})(y_{3s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{3s})-(x_{1s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{1s})$$
$$\cdot\ (y_{2s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{2s})-(x_{3s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{3s})(y_{4s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{4s})\}]$$
$$\cdot\ [\sum_{s=1}^{n}\sum_{r=1}^{4}(x_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}x_{rs})(y_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}y_{rs})]^{-1}\ ,$$

$$\hat\xi_{13} = \hat\xi_{11}[\sum_{s=1}^{n}\{(x_{2s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{2s})(y_{4s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{4s})$$
$$+ (x_{3s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{3s})(y_{1s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{1s})-(x_{1s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{1s})$$
$$\cdot\ (y_{3s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{3s})-(x_{4s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{4s})(y_{2s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{2s})\}]$$
$$\cdot\ [\sum_{s=1}^{n}\sum_{r=1}^{4}(x_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}x_{rs})(y_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}y_{rs})]^{-1}\ ,$$

$$\hat\xi_{14} = \hat\xi_{11}[\sum_{s=1}^{n}\{(x_{3s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{3s})(y_{2s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{2s})$$
$$+ (x_{4s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{4s})(y_{1s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{1s})-(x_{1s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{1s})$$
$$\cdot\ (y_{4s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{4s})-(x_{2s}-\tfrac{1}{n}\sum_{s=1}^{n}x_{2s})(y_{3s}-\tfrac{1}{n}\sum_{s=1}^{n}y_{3s})\}]$$
$$\cdot\ [\sum_{s=1}^{n}\sum_{r=1}^{4}(x_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}x_{rs})(y_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}y_{rs})]^{-1}\ ,$$

$$\hat\sigma_1^2 = \hat\xi_{11}[\sum_{s=1}^{n}\sum_{r=1}^{4}(x_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}x_{rs})^2][\sum_{s=1}^{n}\sum_{r=1}^{4}(x_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}x_{rs})$$
$$\cdot\ (y_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}y_{rs})]^{-1}\ \text{en}$$

$$\hat\sigma_2^2 = \hat\xi_{11}[\sum_{s=1}^{n}\sum_{r=1}^{4}(y_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}y_{rs})^2][\sum_{s=1}^{n}\sum_{r=1}^{4}(x_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}x_{rs})$$
$$\cdot\ (y_{rs}-\tfrac{1}{n}\sum_{s=1}^{n}y_{rs})]^{-1}$$

waaruit (5.3) – (5.6) dan maklik volg.

OPMERKING 5.1

Let op dat indien die beramer vir Σ_0^* volgens die bekende matriksdifferensiasie-tegnieke bepaal is, die beramer vir Σ_0^* gegee word deur $\hat{\Sigma}_0^* = \frac{1}{n}\sum_{s=1}^{n}(\underline{z}_0^*-\hat{\underline{\mu}}_0^*)(\underline{z}_0^*-\hat{\underline{\mu}}_0^*)'$. As gevolg van die spesiale struktuur van Σ_0^* bring dit mee dat dieselfde parameters deur verskillende beramers beraam word. Deur die gemiddelde van hierdie verskillende beramers vir dieselfde parameter te bepaal, lewer dit dieselfde beramers soos in (5.3) tot (5.6).

6. KWATERNIOONHIPOTESE EN -TOETSKRITERIUM

Aan die hand van 'n voorbeeld word twee moontlike maniere uitgewys om 'n toetskriterium te verkry. As gevolg van die feit dat die vermenigvuldiging van kwaternione nie-kommutatief is, is die kwaternioontoetskriteria in moeilike-hanteerbare vorme. In die voorbeeld is daar egter vir die geval waar die kovariansiematriks bekend is, 'n kwaternioontoetskriterium verkry. Alhoewel die toetskriterium in terme van kwaternioonhoeveelhede gedefinieer word, vereenvoudig dit tot 'n gewone reële hoeveelheid. Hieruit volg dat die kwaternioonhipotese in hierdie geval ekwivalent is aan 'n reële hipotese, sodat die vraag nou ontstaan of dit sinvol sal wees om telkens 'n reële toetskriterium te gebruik om die betrokke kwaternioonhipotese te toets.

VOORBEELD 6.1

Veronderstel

$$\underline{Z}: 2\times 1 = \underline{X}_1 + i\underline{X}_2 + j\underline{X}_3 + k\underline{X}_4 \sim \mathrm{QMN}(2;\underline{\mu},\Sigma)\ \text{waarby}$$

$$\underline{\mu}: 2\times 1 = \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} = \underline{\mu}_{x_1} + i\underline{\mu}_{x_2} + j\underline{\mu}_{x_3} + k\underline{\mu}_{x_4}\ \text{en}$$

$$\Sigma : 2\times 2 = \begin{bmatrix} \sigma_1^2 & \xi_{11}+i\xi_{12}+j\xi_{13}+k\xi_{14} \\ \xi_{11}-i\xi_{12}-j\xi_{13}-k\xi_{14} & \sigma_2^2 \end{bmatrix}$$

'n Hermities-positief-definiete matriks is.

Die kwaternioonnulhipotese wat in hierdie voorbeeld beskou word, is

$$H_0 : \underline{\mu} = \underline{0}\,,\ \Sigma \text{ bekend}$$

wat op grond van 'n ewekansige steekproef $\underline{Z}_1, \ldots, \underline{Z}_N$ van $\underline{Z}$ teenoor

$$H_a : \underline{\mu} \neq \underline{0}\,,\ \Sigma \text{ bekend}$$

getoets word.

Om 'n toetskriterium vir hierdie situasie af te lei, beskou die aanneemlikheidsfunksie

$$\begin{aligned} L = {} & (\tfrac{2}{\pi})^{4N}[\sigma_1^2\sigma_2^2-(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2)]^{-2N} \\ & \bullet \text{ eks}\{-2(\sigma_1^2\sigma_2^2-(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2))^{-1} \\ & \bullet [\sigma_2^2 \sum_{s=1}^{N} \overline{(z_{1s}-\mu_1)}(z_{1s}-\mu_1) \\ & - 2RD \sum_{s=1}^{N} \overline{(z_{1s}-\mu_1)}(\xi_{11}+i\xi_{12}+j\xi_{13}+k\xi_{14})(z_{2s}-\mu_2) \\ & + \sigma_1^2 \sum_{s=1}^{N} \overline{(z_{2s}-\mu_2)}(z_{2s}-\mu_2)]\} \end{aligned}$$

van 'n gegewe ewekansige steekproef. Die aanneemlikheidsverhouding-kriterium word gegee deur

$$\Lambda = \frac{\underset{H_0}{\text{maks}}\, L(\underline{\mu},\Sigma)}{\underset{H_a}{\text{maks}}\, L(\underline{\mu},\Sigma)}$$

Onder H_0 volg dus dat

(6.1)

$$\begin{aligned} \underset{H_0}{\text{maks}}\, L(\underline{\mu},\Sigma) = {} & (\tfrac{2}{\pi})^{4N}(\sigma_1^2\sigma_2^2-(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2))^{-2N} \\ & \bullet \text{ eks}\{-2(\sigma_1^2\sigma_2^2-(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2))^{-1} \\ & \bullet [\sigma_2^2 \sum_{s=1}^{N} \bar{z}_{1s}z_{1s} \\ & - 2RD \sum_{s=1}^{N} \bar{z}_{1s}(\xi_{11}+i\xi_{12}+j\xi_{13}+k\xi_{14})z_{2s} \\ & + \sigma_1^2 \sum_{s=1}^{N} \bar{z}_{2s}z_{2s}]\}. \end{aligned}$$

Uit (5.1) en (5.2) volg dag die maksimum-aanneemlikheidsberamer vir $\underline{\mu}$ gegee word deur

$$\hat{\underline{\mu}} = \frac{1}{N}\sum_{s=1}^{N} \underline{z}_s = (\text{gem } z_1, \text{gem } z_2)'$$

sodat

(6.2)

$$\begin{aligned} \underset{H_a}{\text{maks}}\, L(\underline{\mu},\Sigma) = {} & L(\hat{\underline{\mu}},\Sigma) \\ = {} & (\tfrac{2}{\pi})^{4N}(\sigma_1^2\sigma_2^2-(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2))^{-2N} \\ & \bullet \text{ eks}\{-2(\sigma_1^2\sigma_2^2-(\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2))^{-1} \\ & \bullet [\sigma_2^2 \sum_{s=1}^{N} \overline{(z_{1s}-\text{gem} z_1)}(z_{1s}-\text{gem } z_1) \\ & - 2RD \sum_{s=1}^{N} \overline{(z_{1s}-\text{gem } z_1)} \\ & \bullet (\xi_{11}+i\xi_{12}+j\xi_{13}+k\xi_{14})(z_{2s}-\text{gem } z_2) \\ & + \sigma_1^2 \sum_{s=1}^{N} \overline{(z_{2s}-\text{gem } z_2)}(z_{2s}-\text{gem } z_2)]\}. \end{aligned}$$

Uit (6.1) en (6.2) volg dus dat

$$\begin{aligned} \Lambda = {} & \text{eks}\{-2N(\sigma_1^2\sigma_2^2 \\ & - (\xi_{11}^2+\xi_{12}^2+\xi_{13}^2+\xi_{14}^2)^{-1}[\sigma_1^2 \text{gem}\bar{z}_1 \text{ gem} z_1 \\ & - 2RD \text{ gem}\bar{z}_1(\xi_{11}+i\xi_{12}+j\xi_{13}+k\xi_{14})\text{gem} z_2 \\ & + \sigma_2^2 \text{gem}\bar{z}_2 \text{ gem} z_2]\} \\ = {} & \text{eks}\{-2N \text{ gem}\underline{\bar{z}}'\Sigma^{-1}\text{gem}\underline{z}\}\,. \end{aligned}$$

Die nulhipotese H_0 sal nou by die 100%-betekenispeil ten gunste van H_a verwerp word indien

$$\text{eks}\{-2N\text{gem}\underline{\bar{z}}'\Sigma^{-1}\text{gem}\underline{z}\} \leqslant \lambda_\alpha$$

waarby die konstante λ_α sodanig is dat $P(\Lambda \leqslant \lambda_\alpha | H_{01}) = \alpha$. Dus verwerp H_0 indien $y = 4N\text{gem}\underline{\bar{z}}'\Sigma^{-1}\text{gem}\underline{z} \geqslant -2\ell n\lambda_\alpha = \lambda_\alpha^*$.

Onder H_0 is gem$\underline{Z} \sim QMN(2;\underline{0},\frac{1}{n}\Sigma)$ en volgens stelling (4.3)

$$Y = 4N \text{ gem}\underline{\bar{Z}}'\Sigma^{-1}\text{gem}\underline{Z} \sim \chi_8^2.$$

Omdat $P(Y \geqslant \chi_{8,1-\alpha}^2) = \alpha$ word die nulhipotese verwerp indien $y \geqslant \chi_{8,1-\alpha}^2$ waar $\chi_{8,1-\alpha}^2$ die $100(1-\alpha)$ persentiel van χ_8^2 is.

'n Tweede moontlike manier om 'n toetskriterium vir hierdie situasie af te lei is om die geassosieerde reële toevalsvektor van $\underline{Z}$ te beskou; Dit is bekend dat

$$\underline{Z}_0^* : 8\times 1 \sim MN(8;\underline{\mu}_0^*,\Sigma_0^*)$$

waarby $\underline{\mu}_0^* : 8\times 1 = [\mu_{11}, \mu_{21}, \mu_{31}, \mu_{41}, \mu_{12}, \mu_{22}, \mu_{23}, \mu_{24}]'$ en Σ_0^* soos gegee in (5.7).

Die toetskriterium vir die toets van die hipotese

$$H_0^* : \underline{\mu}_0^* = \underline{0},\ \Sigma_0^* \text{ bekend}$$

teenoor

$$H_a^* : \underline{\mu}_0^* \neq \underline{0},\ \Sigma_0^* \text{ bekend}$$

op grond van 'n ewekansige steekproef $\underline{Z}_{01}^*, \ldots, \underline{Z}_{0N}^*$ van $\underline{Z}_0^*$ word gegee deur (vergelyk de Waal en Nel, 1979):

$$Y_0 = N(\text{gem } \underline{Z}_0^{*\prime}\, \Sigma_0^{*-1} \text{ gem } \underline{Z}_0^*) \sim \chi_8^2$$

waar $\text{gem } \underline{Z}_0^* = [\text{gem } X_{11}, \text{gem } X_{21}, \text{gem } X_{31}, \text{gem } X_{41}, \text{gem } X_{12}, \text{gem } X_{22}, \text{gem } X_{32}, \text{gem } X_{42}]'$

sodat die nulhipotese verwerp word indien

$y_0 \geqslant \chi_{8,1-\alpha}^2$ waar $\chi_{8,1-\alpha}^2$ die $100(1-\alpha)$ persentiel van χ_8^2 is.

OPMERKING 6.1

Uit voorafgaande voorbeeld blyk dit duidelik dat twee benaderings gevolg kan word om $H_0 : \underline{\mu} = \underline{0}$ teenoor $H_a : \underline{\mu} \neq 0$ met Σ bekend te toets. Eerstens kan 'n analise in terme van kwaternioonvariate deurgevoer word met Y as die toetskriterium. Tweedens is dit ook moontlik om die analise in terme van die geassosieerde reële variate uit te voer met Y_0 as die toetskriterium. Hieruit kan dus afgelei word dat die kwaternioonhipotese $H_0 : \underline{\mu} = \underline{0}$ teenoor $H_a : \underline{\mu} \neq \underline{0}$ ook uitgedruk kan word in terme van reële hoeveelhede naamlik $H_0^* : \underline{\mu}_0^* = \underline{0}$ teenoor $H_a^* : \underline{\mu}_0^* \neq \underline{0}$.

Vir Σ onbekend, is die verdeling van die matriks $\hat{\Sigma}$ onbekend, sodat dit moeilik is om 'n geskikte toetskriterium in terme van kwaternioonvariate af te lei. Die vraag ontstaan nou of elke kwaternioonhipotese ekwivalent is aan 'n reële hipotese sodat 'n bekende reële toetskriterium telkens gebruik kan word om die betrokke kwaternioonhipotese te toets. Huidiglik is dit moeilik om daaroor 'n uitspraak te gee, maar dit lyk asof die afleiding van 'n kwaternioon-Wishartverdeling (wat op hierdie stadium nog nie gevind kon word nie) 'n moontlike oplossing vir die probleem kan bied.

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